

Properties of Quantum Hall Skyrmions from Anomalies

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Abstract

It is well known that the Fractional Quantum Hall Effect (FQHE) may be effectively represented by a Chern-Simons theory. In order to incorporate QH Skyrmions, we couple this theory to the topological spin current, and include the Hopf term. The cancellation of anomalies for chiral edge states, and the proviso that Skyrmions may be created and destroyed at the edge, fixes the coefficients of these new terms. Consequently, the charge and the spin of the Skyrmion are uniquely determined. For those two quantities we find the values $e\nu N_{Sky}$ and $\nu N_{Sky}/2$, respectively, where e is electron charge, ν is the filling fraction and N_{Sky} is the Skyrmion winding number. We also add terms to the action so that the classical spin fluctuations in the bulk satisfy the standard equations of a ferromagnet, with spin waves that propagate with the classical drift velocity of the electron.

The FQHE admits a Landau-Ginzburg description in terms of a complex doublet of bosonic fields $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and a statistical Chern-Simons gauge field [1]. The Chern-Simons coupling is chosen such that each “bosonized” electron carries an odd number of elementary flux units, yielding fermionic statistics. The Landau-Ginzburg ground state is given by $\psi_1 = \sqrt{\rho_0}$, $\psi_2 = 0$, where the ground state density of electrons ρ_0 is equal to the filling fraction ν times the external magnetic field divided by the elementary flux unit $2\pi/e$. The corresponding wave function, when expressed in position space, is nothing but the Laughlin wave function.

The lowest lying excitations around the ground state are described by the quasiparticle and quasihole Laughlin wave functions. The presence of a Zeeman interaction naively precludes any

dynamics for the spin degrees of freedom, and consequently the quasiparticles and quasiholes are fully polarized. In the Landau-Ginzburg description, those excitations are associated with vortices. More specifically, they are field configurations where the phase of ψ_1 has a nonzero winding, and $\psi_2 = 0$ everywhere. The magnitude of ψ_1 , which is the square root of the density of electrons, vanishes at points associated with the origin of the vortices.

On the other hand, it was noticed [2] that in cases where the gyromagnetic ratio is small, as for example in *GaAs* samples, the spin degrees of freedom play a dynamical role, and lowest lying excitations above the ground state have some “ferromagnetic” properties[3]. In the Landau-Ginzburg description, this situation is realized by allowing for a nonzero ψ_2 in the spatial domain, which for the moment we define to be all of \mathbf{R}^2 . If we also assume that the number density $\Psi^\dagger\Psi$ never vanishes on \mathbf{R}^2 , it is possible to everywhere identify an $SU(2)$ field degree of freedom g , associated with spin fluctuations. A $U(1)$ subgroup (which we take to be generated by the third Pauli matrix σ_3) of $SU(2)$ is gauged via the coupling to a statistical gauge field (the same gauge field mentioned above), so that the gauge invariant observables are defined on S^2 . Energy finiteness generally demands that g goes to the above $U(1)$ subgroup, i.e. $g \rightarrow \exp i\chi\sigma_3$, at spatial infinity. This corresponds to Ψ going to the ground state value at spatial infinity, and in effect compactifies \mathbf{R}^2 to S^2 . Skyrmions[4],[3] associated with $\Pi_2(S^2)$ result, the elements of $\Pi_2(S^2)$ being labeled by the winding number

$$N_{Sky} = \int_{\mathbf{R}^2} d^2x T^0(g) = \frac{i}{4\pi} \int_{\mathbf{R}^2} d \text{Tr} \sigma_3 g^\dagger dg , \quad (1)$$

where $T^0(g)$ is the time component of the topological current,

$$T^\mu(g) = \frac{i}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \sigma_3 \partial_\nu g^\dagger \partial_\lambda g . \quad (2)$$

(The topological current can be defined for vortices as well, only there it becomes singular at the zeros of $\Psi^\dagger\Psi$.)

A description dual to the Landau-Ginzburg model was found useful for the analysis of vortex dynamics [5]. It is of interest to write down a dual description suitable for the analysis of Skyrmion dynamics as well [6],[7]. This is one of the purposes of our letter. In this regard, we shall argue in favor of including a Hopf term in the action. We shall determine its coefficient, as well as all coefficients, by requiring the model, including its edge terms to be anomaly free [8]. These coefficients then fix Skyrmion properties, such as charge and spin. Arguments have been given in the literature which show that the charge is equal to the winding number times ν times the electron charge[3], which we shall confirm. Although there is some debate, it is generally agreed that the spin should be 1/2 for Skyrmions with winding number one at $\nu = 1$ [9]. We shall confirm this result as well.

We first show what are the consequences of not including a Hopf term. Our starting point is the bulk action [12]

$$\mathcal{S}_H = \int_{\Sigma \times R^1} d^3x \left(\frac{\sigma_H}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - e A_\mu T^\mu \right) , \quad (3)$$

where Σ is the two dimensional spatial domain of the sample, R^1 accounts for time, and $e\mathcal{T}^\mu$ is the Skyrmin current. Additional terms will be added, but for the moment we just consider (3). For us, A_μ is the external electromagnetic field which is not a dynamical variable. Its variations therefore just define the bulk current J_{em}^μ by $J_{em}^\mu = -\frac{\delta\mathcal{S}_H}{\delta A_\mu}$.

According to (3), the bulk electromagnetic current J_{em}^μ is,

$$J_{em}^\mu = -\frac{\sigma_H}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} + e\mathcal{T}^\mu. \quad (4)$$

For consistency the current J_{em}^μ , and consequently \mathcal{T}^μ , must be conserved. This is the case for \mathcal{T}^μ proportional to the topological current T^μ [10],[11]:

$$\mathcal{T}^\mu = \kappa T^\mu. \quad (5)$$

Eq. (4) implies that the electric charge density is $J_{em}^0 = -\sigma_H F_{12} + e\kappa T^0$. Integrating it over the whole sample gives the total electric charge as $-eN_{el} + e\kappa N_{Sky}$, where N_{el} is the total number of electrons at filling fraction ν^* . Thus κ times e is the charge of a Skyrmin of unit winding number.

We now examine under what conditions the bulk action \mathcal{S}_H is consistent with the existence of chiral edge currents. For the case of filling fraction $\nu = 1$, there is a single edge current on the boundary $\partial\Sigma$ of Σ , which may be represented by a 2d massless chiral relativistic Dirac fermion [12] [11], while for fractional values of ν one gets a Luttinger liquid. Chirality implies that the electromagnetic current J_{em}^μ of the edge fermions satisfies $J_-^{em} = \frac{1}{\sqrt{2}}(J_{em}^0 + J_{em}^1) = 0$. In the quantum theory this is known to lead to an anomaly, i.e. $\partial_\mu J_{em}^\mu \neq 0$.

It is convenient to bosonize the edge theory [13], and for this we shall introduce a scalar field ϕ on $\partial\Sigma$. In terms of this field, chirality will mean the following:

$$\mathcal{D}_-\phi = f(x^-), \quad (6)$$

where $x^- = \frac{1}{\sqrt{2}}(x^0 - x^1)$, $\mathcal{D}_\pm = \frac{1}{\sqrt{2}}(\mathcal{D}_0 \pm \mathcal{D}_1)$ and \mathcal{D}_μ denotes a covariant derivative. (Usually the more restrictive condition $f(x^-) = 0$ is assumed, but (6) seems enough for us.)

To proceed we shall pose an action principle for the edge field ϕ . The edge action $\mathcal{S}_{\partial\Sigma \times R^1}$ should be such that: i) The total action $\mathcal{S} = \mathcal{S}_H + \mathcal{S}_{\partial\Sigma \times R^1}$ is gauge invariant. ii) It is consistent with chirality, i.e. (6). We will show that these two conditions lead to a chiral electromagnetic current $J_-^{em} = 0$, which at the boundary is defined by $J_{em}^\mu = -\frac{\delta\mathcal{S}}{\delta A_\mu}|_{\partial\Sigma \times R^1}$. Requirements i) and ii) also lead to the anomaly. For this recall that the one loop effects responsible for the anomaly in the fermionic theory appear at tree level in the bosonized theory. Thus we can expect to recover the anomaly from the classical equation of motion for ϕ .

We begin by addressing the issue i) of gauge invariance. If we ignore boundary effects, the bulk action is separately gauge invariant under transformations of the electromagnetic potentials A_μ ,

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (7)$$

*Here we have used the result $\sigma_H = \frac{e^2\nu}{2\pi}$.

as well as under transformations of the fields g ,

$$g \rightarrow g e^{i\lambda\sigma_3}, \quad (8)$$

where both Λ and λ are functions of space-time coordinates. On the other hand, taking into account the boundary $\partial\Sigma$, one finds instead that (7) gives the surface terms

$$\delta\mathcal{S}_H = -\frac{\sigma_H}{2} \int_{\partial\Sigma \times R^1} d\Lambda \wedge A + \frac{e\kappa i}{4\pi} \int_{\partial\Sigma \times R^1} d\Lambda \wedge \text{Tr}\sigma_3 g^\dagger dg, \quad (9)$$

while gauge invariance under transformations (8) persists. We now specify that under gauge transformations (7), the edge field ϕ transforms according to

$$\phi \rightarrow \phi + e\Lambda. \quad (10)$$

Then we can cancel both of the above boundary terms in (9) if we assume the following action for the scalar field ϕ :

$$\mathcal{S}_{\partial\Sigma \times R^1} = \frac{R^2}{8\pi} \int_{\partial\Sigma \times R^1} d^2x (\mathcal{D}_\mu \phi)^2 + \frac{\sigma_H}{2e} \int_{\partial\Sigma \times R^1} d\phi \wedge A - \frac{\kappa i}{4\pi} \int_{\partial\Sigma \times R^1} d\phi \wedge \text{Tr}\sigma_3 g^\dagger dg. \quad (11)$$

In (11) we have added a kinetic energy term for ϕ , where the covariant derivative is defined by $\mathcal{D}_\mu \phi = \partial_\mu \phi - eA_\mu$. The coefficient R is real and is known to correspond to the square root of the filling fraction ν . In this regard, a straightforward quantization of the edge Lagrangian shows that R^2 is the ratio of odd integers, and more generally, that entire hierarchies of filling fractions can be obtained[13].

Concerning ii), extremizing (11) with respect to ϕ gives

$$R^2 \partial_\mu \mathcal{D}^\mu \phi = -\frac{2\pi\sigma_H}{e} F_{01} - 4\pi \mathcal{T}^r, \quad (12)$$

F_{01} being the electric field strength at the boundary and the index r denoting the direction normal to the surface. This equation can be rewritten as

$$2R^2 \partial_+ \mathcal{D}_- \phi = (eR^2 - \frac{2\pi\sigma_H}{e}) F_{01} - 4\pi \mathcal{T}^r, \quad (13)$$

using $\partial_+ = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1)$ and $diag(1, -1)$ for the Lorentz metric. But the chirality condition (6) requires that the right hand side of (13) vanishes. For this we can set

$$\sigma_H = \frac{e^2 R^2}{2\pi}, \quad (14)$$

which is the usual relation for the Hall conductivity (after identifying R^2 with the filling fraction ν). But we also need

$$\mathcal{T}^r = 0 \quad \text{at } \partial\Sigma. \quad (15)$$

From (14) and (15), variations of A_μ give the following result for the edge current

$$J_{em}^\mu = -\frac{\delta\mathcal{S}}{\delta A_\mu} |_{\partial\Sigma \times R^1} = \frac{eR^2}{4\pi} (\mathcal{D}^\mu + \epsilon^{\mu\nu} \mathcal{D}_\nu) \phi, \quad (16)$$

and thus it is chiral, i.e. $J_-^{em} = 0$. Here $\epsilon^{01} = -\epsilon^{10} = 1$. By taking its divergence we also recover the anomaly:

$$\partial_\mu J_{em}^\mu = \frac{eR^2}{4\pi} \partial_\mu (\mathcal{D}^\mu + \epsilon^{\mu\nu} \mathcal{D}_\nu) \phi = -\frac{e^2 R^2}{2\pi} F_{01} , \quad (17)$$

where we again used (14) and (15).

In order to satisfy chirality in the above discussion, we needed not only to constrain the values of coefficients, but we also found it necessary to impose a boundary condition (15) on the topological current. As a result, the topological flux, and moreover Skyrmions, cannot penetrate the edge. Thus, provided g is everywhere defined in Σ , the total Skyrmion number within the bulk $\int_\Sigma d^2x T^0(g)$ is a conserved quantity, and for example, a nonzero value for the total topological charge cannot be adiabatically generated from the ground state.

Below, we generalize to the situation where the total Skyrmion number in the bulk is *not* restricted to being a constant. For this we need to drop the boundary condition (15), and thus allow for a nonzero topological flux into or out of the sample. One may interpret this as Skyrmions being created or destroyed at the edges.[†] For this purpose, we consider an extension of the above description, where the Hopf term

$$\mathcal{S}_{WZ} = \frac{\Theta}{24\pi^2} \int_{\Sigma \times R^1} \text{Tr}(g^\dagger dg)^3 \quad (18)$$

is added to the bulk action \mathcal{S}_H . [Note that (18) is a local version of the Hopf term]. This term does not affect the classical equations of motion since it is the integral of a closed three form. It is not well defined for vortices, and must be suitably regularized in that case. On the other hand, the Hopf term is well defined for Skyrmions. Its utility is in the fact that it provides a direct way for computing the Skyrmion's intrinsic spin, as we now show. Let $g^{(N_{Sky},0)}(\vec{x})$ denote a static field configuration which is nontrivial in a spatial domain $\mathcal{V} \subset \Sigma$ and goes to $\exp i\chi\sigma_3$ at the boundary $\partial\mathcal{V}$ of this region. More generally, we can define a one parameter family of configurations, using

$$g^{(N_{Sky},\theta)}(\vec{x}) = e^{i\theta\sigma_3/2} g^{(N_{Sky},0)}(\vec{x}) e^{-i\theta\sigma_3/2} , \quad (19)$$

which corresponds to a spin rotation by an angle θ . The gauge choice made in (19) is such that the vacuum values (which have the form $e^{i\chi\sigma_3}$) of $g^{(N_{Sky},\theta)}(\vec{x})$ are invariant under rotations.[14] (This fact was neglected in [15].) As a result of this choice, $g^{(N_{Sky},\theta)}$ evaluated at the edge does not depend on θ , and hence the edge action is unaffected by such rotations. Now we consider an adiabatic rotation by 2π . Thus we set $\theta = \theta(t)$, with $\theta(-\infty) = 0$ and $\theta(\infty) = 2\pi$. To get the spin we compute the classical bulk action, specifically \mathcal{S}_{WZ} , for this process. We get

$$\begin{aligned} \mathcal{S}_{WZ} &= \int_{-\infty}^{\infty} dt \int_{\mathcal{V}} d^2x \mathcal{L}_{WZ} = \frac{i\Theta}{8\pi} \int_{\mathcal{V}} \text{Tr} \sigma_3 \left[(dg^{(N_{Sky},0)} g^{(N_{Sky},0)\dagger})^2 - (g^{(N_{Sky},0)\dagger} dg^{(N_{Sky},0)})^2 \right] \\ &= \frac{i\Theta}{8\pi} \int_{\partial\mathcal{V}} \text{Tr} \sigma_3 \left[dg^{(N_{Sky},0)} g^{(N_{Sky},0)\dagger} + g^{(N_{Sky},0)\dagger} dg^{(N_{Sky},0)} \right] \end{aligned}$$

[†]Viewing vortices as punctures or holes in Σ , they too then act as sinks and sources of topological flux.

$$\begin{aligned}
&= \frac{i\Theta}{4\pi} \int_{\partial\mathcal{V}} \text{Tr} \sigma_3 g^{(N_{Sky},0)\dagger} dg^{(N_{Sky},0)} \\
&= \Theta \int_{\mathcal{V}} d^2x T^0(g^{(N_{Sky},0)}) \equiv \Theta N_{Sky} ,
\end{aligned} \tag{20}$$

where we have used Stoke's theorem. Since the action changes by Θ times the winding number under a 2π rotation, its spin (up to an integer) is

$$\frac{\Theta N_{Sky}}{2\pi} . \tag{21}$$

We thus need the numerical value of Θ to determine the spin. For this purpose we now reexamine the boundary dynamics taking into account the Hopf term. We once again require i) gauge invariance and ii) chirality.

Concerning i), as before, the bulk action is not invariant under gauge transformations (7) of the electromagnetic potentials A_μ . In addition, unlike before, it is not invariant under gauge transformations (8) of the fields g . From \mathcal{S}_{WZ} we pick up the surface term

$$\delta\mathcal{S}_{WZ} = \frac{i\Theta}{8\pi^2} \int_{\partial\Sigma \times R^1} d\lambda \wedge \text{Tr} \sigma_3 g^\dagger dg . \tag{22}$$

To cancel this variation along with (9), we once again assume the existence of an edge field ϕ which transforms like (10), simultaneously with the gauge transformations (7) of the electromagnetic potentials A_μ . We further specify that ϕ transforms according to

$$\phi \rightarrow \phi + \lambda , \tag{23}$$

simultaneously with the gauge transformations (8) of the fields g . Then we can cancel both of the boundary terms (9) and (22), making our theory anomaly free, if we assume the following action for the scalar field ϕ :

$$\begin{aligned}
\mathcal{S}_{\partial\Sigma \times R^1} &= \frac{R^2}{8\pi} \int_{\partial\Sigma \times R^1} d^2x (\mathcal{D}_\mu \phi)^2 + \frac{\sigma_H}{2e} \int_{\partial\Sigma \times R^1} d\phi \wedge A \\
&\quad - \frac{i}{4} \int_{\partial\Sigma \times R^1} \left(\frac{\Theta}{2\pi^2} d\phi + \frac{\sigma_H}{e} A \right) \wedge \text{Tr} \sigma_3 g^\dagger dg ,
\end{aligned} \tag{24}$$

provided we also impose that

$$\kappa = \frac{\pi\sigma_H}{e^2} + \frac{\Theta}{2\pi} . \tag{25}$$

Since ϕ admits gauge transformations (23), as well as (10), we must redefine the covariant derivative appearing in (24) according to

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - \beta_\mu , \quad \beta_\mu = eA_\mu - \frac{i}{2} \text{Tr} \sigma_3 g^\dagger \partial_\mu g . \tag{26}$$

With regard to ii), the equation of motion for ϕ is

$$R^2 \partial_\mu \mathcal{D}^\mu \phi = -\frac{2\pi\sigma_H}{e} F_{01} - 2\Theta T^r , \tag{27}$$

which can be rewritten as

$$2R^2\partial_+\mathcal{D}_-\phi = (eR^2 - \frac{2\pi\sigma_H}{e})F_{01} + 2(\pi R^2 - \Theta)\mathcal{T}^r . \quad (28)$$

We recover the chirality condition (6) upon setting

$$\Theta = \pi R^2 , \quad (29)$$

as well as (14). From (14) and (29), variations of A_μ again give the the edge current in the form of (16) (although the covariant derivative is now defined differently) and thus $J_-^{em} = 0$. By taking its divergence we get the anomaly equation:

$$\partial_\mu J_{em}^\mu = -\frac{eR^2}{2\pi}\epsilon^{\mu\nu}\partial_\mu\beta_\nu . \quad (30)$$

Thus now we can satisfy the criterion of chirality without imposing any boundary conditions on the topological current. Substituting (29) into (25) (and using $R^2 = \nu$) also fixes κ to be the filling fraction. It follows that the Skymion charge is $e\nu N_{Sky}$. Eqs. (21) and (29) then give the value for the spin to be $\frac{N_{Sky}\nu}{2}$. Therefore, within the above assumptions, a winding number one Skymion is a fermion when the filling fraction is one.

We note further that using the above values for the constants, we can simplify the bulk action $\mathcal{S}_H + \mathcal{S}_{WZ}$ to the single Chern-Simons term

$$\frac{\nu}{4\pi} \int_{\Sigma \times R^1} \beta \wedge d\beta \quad (31)$$

plus the surface term

$$\frac{ie\nu}{8\pi} \int_{\partial\Sigma \times R^1} A \wedge \text{Tr}\sigma_3 g^\dagger dg . \quad (32)$$

This surface term cancels the last term in (24), and consequently the boundary action simplifies to

$$\frac{\nu}{8\pi} \int_{\partial\Sigma \times R^1} d^2x (\partial_\mu\phi - \beta_\mu)^2 + \frac{\nu}{4\pi} \int_{\partial\Sigma \times R^1} d\phi \wedge \beta . \quad (33)$$

Thus if no additional terms are present, and there are no vortex singularities present in the sample, the bulk plus edge action can be written entirely in terms of the redefined Chern-Simons connection β_μ and the edge field ϕ .

In the above treatments we have just considered the topological sector of our dual description. We have not incorporated the terms responsible for spin fluctuations in the bulk, and other possible higher derivative terms. It is important to have in mind that there are spin waves, the Goldstone modes associated with a ferromagnetic ground state. We will include those fluctuations by demanding that the classical equations of motion are those of a ferromagnet, and have Skyrmons among its solutions. That is achieved by including a kinematic term for the Skymion:

$$\mathcal{S}_S = -\frac{\eta}{2} \int_{\Sigma \times R^1} d^3x (\partial_i n_a)^2 , \quad a = 1, 2, 3 , \quad (34)$$

where n_a is a unit vector field defined by $n_a \sigma_a = g \sigma_3 g^\dagger$ and η is a constant. Now, the dynamics in the bulk cannot be described solely by the Chern-Simons connection β_μ . Starting from the total bulk action $\mathcal{S}_H + \mathcal{S}_{WZ} + \mathcal{S}_S$, the dynamics for the spin degrees of freedom is readily obtained from infinitesimal variations $\delta g = i \epsilon_a \sigma_a g$ of g :

$$2\eta(\nabla^2 n \times n)_a = \frac{e\nu}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\mu n_a \partial_\nu A_\lambda . \quad (35)$$

which, we can rewrite as,

$$\partial_0 n_a - v_j \partial_j n_a - \frac{2\eta}{\rho_0} (\nabla^2 n \times n)_a = 0 , \quad (36)$$

where ρ_0 is the ground state density of electrons, \vec{v} is their classical drift velocity (which is in the plane of the sample and perpendicular to the electric field E , with total velocity $|v| = cE/B$), and the first two terms denote the convective derivative of n_a . This is the same equation that follows from the Landau-Ginzburg formalism [6] as well.

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